

Integrable (2+1)-dimensional spin models with self-consistent potentials

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Abstract

Integrable spin systems possess interesting geometrical and gauge invariance properties and have important applications in applied magnetism and nanophysics. They are also intimately connected to nonlinear Schrödinger family of equations. In this paper we identify three different integrable spin systems in (2+1) dimensions by introducing interaction of the spin field with more than one scalar potentials or vector potential or both. We also obtain the associated Lax pairs. We discuss various interesting reductions in (2+1) and (1+1) dimensions. We also deduce the equivalent nonlinear Schrödinger family of equations, including (2+1) dimensional version of nonlinear Schrödinger-Hirota-Maxwell-Bloch equations, along with their Lax pairs.

1 Introduction

Integrable and nonintegrable spin systems [1] play a very useful role in nonlinear physics and mathematics. They give rise to important applications in applied magnetism [2] and nanophysics [3]. The Landau-Lifshitz-Gilbert (LLG) equation [4] in ferromagnetism and Landau-Lifshitz-Gilbert-Slonczewski (LLGS) equation [3] in spin transfer nanomagnetic multilayers are some of the fundamental equations which play a crucial role in understanding various physical properties of magnetic materials and the development of new technological innovations like microwave generation using spin transfer effect [5]. The continuum limit of the Heisenberg ferromagnetic spin system and its various generalizations give rise to some of the important integrable spin systems in (1+1) dimensions [6, 7]. They are also intimately related to the nonlinear Schrödinger family of equations through geometrical (or Lakshmanan equivalence or L-equivalence) and gauge equivalence concepts and these systems often admit magnetic soliton solutions [1].

Though a straightforward generalization of the (1+1) dimensional Heisenberg spin system to (2+1) dimensions is not integrable [8], inclusion of additional terms corresponding to interaction of a scalar potential field makes the spin system integrable. The well known Ishimori equation [9] and Myrzakulov I equation [10] are two of the most interesting integrable spin equations. Their geometrical and gauge equivalent counterparts are the Davey-Stewartson and Strachan equations [11] respectively. They admit (2+1) dimensional localized structures [11, 12]. Interestingly such an interaction of the spin vector with the scalar potential can be further generalized. One can include more than one scalar potential and make them to interact with the spin vector to generate new integrable spin equations. Furthermore one can even introduce the interaction of a vector (unit) potential with the spin vector. The result is that one can obtain more general integrable (2+1) dimensional spin evolution equations along with their associated Lax pairs. In this paper, we introduce three such integrable spin models in (2+1) dimensions, namely Myrzakulov-Lakshmanan (ML) II, III and IV equations. We also point out that equivalent (2+1) dimensional integrable nonlinear Schrödinger-Maxwell-Bloch type evolution equations and their Lax pairs can also be

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identified. From these equations several interesting limiting cases of nonlinear evolution equations in (2+1) and (1+1) dimensions, along with their Lax pairs, can also be deduced. In this paper, we do not attempt to explicitly solve the initial value problem associated with the Lax pair obtain explicit localized solutions, which will be reported in subsequent works.

The plan of the paper is as follows. In Sec. II, we give the basic facts from the theory of the generalization of the Heisenberg ferromagnetic spin equation in (2+1) dimensions. In Sec. III, we investigate the ML-II equation. Next, we study the ML-III equation in Sec. IV. In Sec. V we consider the ML-IV equation. Finally, we give our conclusions in Sec. VI.

2 Brief review on integrable spin systems in 2+1 dimensions

There exists a few integrable spin systems in (2+1) dimensions [9, 10, 11, 12, 13]. In this section we present some basic features of them.

2.1 The Ishimori equation

The well known Ishimori equation has the form [9]

$$\mathbf{S}_t - \mathbf{S} \wedge (\mathbf{S}_{xx} + \mathbf{S}_{yy}) - u_x \mathbf{S}_y - u_y \mathbf{S}_x = 0, \quad (2.1)$$

$$u_{xx} - \alpha^2 u_{yy} + 2\alpha^2 \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) = 0, \quad (2.2)$$

where \wedge denotes the vector product (or the cross product), $\alpha = \text{const}$, \mathbf{S} is the spin vector with unit length, that is

$$\mathbf{S} = (S_1, S_2, S_3), \quad \mathbf{S}^2 = 1, \quad (2.3)$$

and u is a scalar real function (potential). The Ishimori equation admits the following Lax representation [5],

$$\Phi_x + \alpha S \Phi_y = 0, \quad (2.4)$$

$$\Phi_t - A_2 \Phi_{xx} - A_1 \Phi_x = 0, \quad (2.5)$$

where

$$A_2 = -2iS, \quad (2.6)$$

$$A_1 = -iS_x - i\alpha S_y S + u_y I - \alpha^3 u_x S. \quad (2.7)$$

Here

$$S = S_i \sigma_i = \begin{pmatrix} S_3 & S^+ \\ S^- & -S_3 \end{pmatrix}, \quad (2.8)$$

where $S^2 = I$, $S^\pm = S_1 \pm iS_2$, $[A, B] = AB - BA$, $I = \text{diag}(1, 1)$ is the identity matrix and the σ_i are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.9)$$

The Ishimori equation is one of the integrable (2+1)-dimensional extensions of the following celebrated integrable (1+1)-dimensional continuum Heisenberg ferromagnetic spin equation (HFE) [6, 7],

$$\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xx} = 0. \quad (2.10)$$

Note that the gauge/geometric equivalent counterpart of the Ishimori equation is the Davey-Stewartson equation [11] which reads as

$$i\varphi_t + \alpha^2 \varphi_{xx} + \varphi_{yy} - v\varphi + 2|\varphi|^2 \varphi = 0, \quad (2.11)$$

$$v_{xx} - \alpha^2 v_{yy} + 4(|\varphi|^2)_{yy} = 0. \quad (2.12)$$

It is one of the (2+1)-dimensional integrable extensions of the nonlinear Schrödinger equation (NSE)

$$i\varphi_t + \varphi_{xx} + 2|\varphi|^2 \varphi = 0. \quad (2.13)$$

Different properties of the Ishimori and Davey-Stewartson equations are well studied in the literature [9, 11, 12]. Also we can recall that between the HFE (2.10) and NSE (2.13) takes place the Lakshmanan and gauge equivalence [1, 6, 7].

2.2 The Myrzakulov-I equation

As the second example of the integrable spin systems in (2+1) dimensions, we here present some details of the Myrzakulov-I equation or shortly the M-I equation [13]. It reads as

$$\mathbf{S}_t - \mathbf{S} \wedge \mathbf{S}_{xy} - u\mathbf{S}_x = 0, \quad (2.14)$$

$$u_x + \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) = 0. \quad (2.15)$$

Often we write this equation in the following form

$$\mathbf{S}_t - (\mathbf{S} \wedge \mathbf{S}_y + u\mathbf{S})_x = 0, \quad (2.16)$$

$$u_x + \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) = 0. \quad (2.17)$$

The M-I equation has the following Lax representation

$$\Phi_x - \frac{i}{2}\lambda S\Phi = 0, \quad (2.18)$$

$$\Phi_t - \lambda\Phi_y - \lambda Z\Phi = 0, \quad (2.19)$$

where

$$Z = \frac{1}{4}([S, S_y] + 2iuS). \quad (2.20)$$

As the Ishimori equation, the M-I equation is one of the (2+1)-dimensional extensions of the (1+1)-dimensional HFE (2.10). The gauge/geometric equivalent counterpart of the M-I equation has the form [13]

$$i\varphi_t + \varphi_{xy} + v\varphi = 0, \quad (2.21)$$

$$v_x - 2(|\varphi|^2)_y = 0, \quad (2.22)$$

which is nothing but one of the (2+1)-dimensional extensions of the NSE (2.13) [12].

2.3 The Myrzakulov-Lakshmanan-I equation

Another example of the integrable spin systems in (2+1) dimensions is the so-called Myrzakulov-Lakshmanan-I (ML-I) equation [31] which reads as

$$\mathbf{S}_t - \mathbf{S} \wedge (\alpha S_{xx} + \beta \mathbf{S}_{xy}) - u\mathbf{S}_x = 0, \quad (2.23)$$

$$u_x + \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) = 0. \quad (2.24)$$

It has the following Lax representation

$$\Phi_x - \frac{i}{2}\lambda S\Phi = 0, \quad (2.25)$$

$$\Phi_t - \lambda\Phi_y - B\Phi = 0, \quad (2.26)$$

where

$$B = \alpha(\frac{1}{2}i\lambda^2 S + \frac{1}{4}[S, S_x]) + \beta\lambda Z. \quad (2.27)$$

The Myrzakulov-Lakshmanan-I equation is another integrable (2+1)-dimensional extension of the (1+1)-dimensional Heisenberg ferromagnet equation (2.10). The equivalent counterpart of the Myrzakulov-Lakshmanan-I equation is the evolution equation

$$i\varphi_t + \alpha^2\varphi_{xx} + \beta\varphi_{xy} + v\varphi = 0, \quad (2.28)$$

$$v_x - 2[\alpha(|\varphi|^2)_x + \beta(|\varphi|^2)_y] = 0, \quad (2.29)$$

which is also one of the integrable (2+1)-dimensional extensions of the nonlinear Schrödinger equation (2.13).

2.4 The (2+1)-dimensional Heisenberg ferromagnet equation

The physically important (2+1)-dimensional Heisenberg ferromagnet equation looks like

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \mathbf{S}_{yy}). \quad (2.30)$$

It is a very important from the physical applications point of view equation. But unfortunately this equation is not integrable [8].

3 The Myrzakulov-Lakshmanan-II equation

In this and next two sections we will present a new class of integrable spin systems in (2+1) dimensions. We will give their Lax representations, equivalent counterparts and some reductions. We start from the so-called Myrzakulov-Lakshmanan-II equation (shortly the ML-II equation) which has the form

$$iS_t + \frac{1}{2}([S, S_y] + 2iuS)_x + \frac{1}{\omega}[S, W] = 0, \quad (3.1)$$

$$u_x I - \frac{i}{2}S \cdot [S_x, S_y] = 0, \quad (3.2)$$

$$iW_x + \omega[S, W] = 0 \quad (3.3)$$

or equivalently

$$iS_t + \frac{1}{2}[S, S_{xy}] + uS_x + \frac{1}{\omega}[S, W] = 0, \quad (3.4)$$

$$u_x - \frac{i}{4}\text{tr}(S \cdot [S_x, S_y]) = 0, \quad (3.5)$$

$$iW_x + \omega[S, W] = 0. \quad (3.6)$$

Here $S = S_i \sigma_i$ and $W = W_i \sigma_i$, $i = 1, 2, 3$. The vector $\vec{W} = (W_1, W_2, W_3)$ may be considered as vector potential. The ML-II equation is integrable. In the next subsections we give some important informations of this equation.

3.1 Reductions

Some comments on the reduction of the ML-II equation are in order. First we note that if we put $W = 0$ then the ML-II equation (3.1)-(3.3) reduces to the M-I equation (2.14)-(2.15). If we consider the case $y = x$ then the ML-II equation (3.1)-(3.3) transforms to the following M-XCIX equation [25]

$$iS_t + \frac{1}{2}[S, S_{xx}] + \frac{1}{\omega}[S, W] = 0, \quad (3.7)$$

$$iW_x + \omega[S, W] = 0. \quad (3.8)$$

So the ML-II equation is one of the (2+1)-dimensional integrable extensions of the M-XCIX equation. We also expect the last equation to admit other integrable extensions in (2+1) dimensions.

3.2 Lax representation

The ML-II equation (3.1)-(3.3) is integrable by IST. The corresponding Lax representation can be written in the form

$$\Phi_x = U\Phi, \quad (3.9)$$

$$\Phi_t = 2\lambda\Phi_y + V\Phi. \quad (3.10)$$

Here the matrix operates U and V have the form

$$U = -i\lambda S, \quad (3.11)$$

$$V = \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W, \quad (3.12)$$

where

$$V_1 = Z = \frac{1}{4}([S, S_y] + 2iuS), \quad (3.13)$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \quad (3.14)$$

3.3 Gauge equivalent counterpart of the ML-II equation

Let us find the gauge equivalent counterpart of the ML-II equations (3.1)-(3.3). It is not difficult to verify that the gauge equivalent counterpart of the ML-II equation is given by

$$q_t + \frac{\kappa}{2i}q_{xy} + ivq - 2p = 0, \quad (3.15)$$

$$r_t - \frac{\kappa}{2i}r_{xy} - ivr - 2k = 0, \quad (3.16)$$

$$\frac{i}{2}v_x - \frac{\kappa}{2i}(rq)_y = 0, \quad (3.17)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (3.18)$$

$$k_x + 2i\omega k - 2\eta r = 0, \quad (3.19)$$

$$\eta_x + rp + kq = 0, \quad (3.20)$$

where q, r, p, k are some complex functions, v, η are potential functions, $\kappa = \text{const}$. We call this equation as the (2+1)-dimensional nonlinear Schrödinger-Maxwell-Bloch equation (NSMBE) as in 1+1 dimensions it reduces to the (1+1)-dimensional nonlinear Schrödinger-Maxwell-Bloch equation (see e.g. [36]-[37] and references therein). Of course this equation is also integrable by IST. The corresponding Lax representation to (3.15)-(3.20) reads as

$$\Psi_x = A\Psi, \quad (3.21)$$

$$\Psi_t = \kappa\lambda\Psi_y + B\Psi, \quad (3.22)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (3.23)$$

$$B = B_0 + \frac{i}{\lambda + \omega}B_{-1}. \quad (3.24)$$

Here

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (3.25)$$

$$B_0 = -\frac{i}{2}v\sigma_3 - \frac{\kappa}{2i}\begin{pmatrix} 0 & q_y \\ r_y & 0 \end{pmatrix}, \quad (3.26)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \quad (3.27)$$

If we set $\kappa = 2$ then the system (3.15)-(3.20) takes the form

$$iq_t + q_{xy} - vq - 2ip = 0, \quad (3.28)$$

$$ir_t - r_{xy} + vr - 2ik = 0, \quad (3.29)$$

$$v_x + 2(rq)_y = 0, \quad (3.30)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (3.31)$$

$$k_x + 2i\omega k - 2\eta r = 0, \quad (3.32)$$

$$\eta_x + rp + kq = 0. \quad (3.33)$$

Finally let us we consider the reduction $r = \delta q^*$, $k = \delta p^*$, where $*$ means complex conjugate. Then the system (3.28)-(3.33) takes the form

$$iq_t + q_{xy} - vq - 2ip = 0, \quad (3.34)$$

$$v_x + 2\delta(|q|^2)_y = 0, \quad (3.35)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (3.36)$$

$$\eta_x + \delta(q^*p + p^*q) = 0, \quad (3.37)$$

where we have assumed that $\delta = \pm 1$. We note that in (1+1) dimensions, that is if $y = x$, the last system takes the form

$$iq_t + q_{xx} + 2\delta|q|^2q - 2ip = 0, \quad (3.38)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (3.39)$$

$$\eta_x + \delta(q^*p + p^*q) = 0, \quad (3.40)$$

which is nothing but the well-known (1+1)-dimensional nonlinear Schrödinger-Maxwell-Bloch equation (see e.g. [36]-[37] and references therein). Its Lax pair has the form

$$\Psi_x = A\Psi, \quad (3.41)$$

$$\Psi_t = 2\lambda A\Psi + B\Psi, \quad (3.42)$$

where A and B have the form (3.23)-(3.24) with

$$A_0 = \begin{pmatrix} 0 & q \\ -\delta q^* & 0 \end{pmatrix}, \quad (3.43)$$

$$B_0 = i\delta|q|^2\sigma_3 + i\begin{pmatrix} 0 & q_x \\ \delta q_x^* & 0 \end{pmatrix}, \quad (3.44)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -\delta p^* & -\eta \end{pmatrix}. \quad (3.45)$$

Note that the spin equivalent counterpart of the system (3.38)-(3.40) is given by

$$iS_t + \frac{1}{2}[S, S_{xx}] + \frac{1}{\omega}[S, W] = 0, \quad (3.46)$$

$$iW_x + \omega[S, W] = 0. \quad (3.47)$$

It is nothing but the (1+1)-dimensional M-XCIX equation (3.7)-(3.8) which is well-known to be integrable [25].

4 The Myrzakulov-Lakshmanan-III equation

Now we want to present another new integrable spin system in 2+1 dimensions, namely the so-called the Myrzakulov-Lakshmanan-III (ML-III) equation. It looks like

$$iS_t + i\epsilon_2(S_{xy} + [S_x, Z])_x + (wS)_x + \frac{1}{\omega}[S, W] = 0, \quad (4.1)$$

$$u_x - \frac{i}{4}\text{tr}(S \cdot [S_x, S_y]) = 0, \quad (4.2)$$

$$w_x - \frac{i}{4}\epsilon_2[\text{tr}(S_x^2)]_y = 0, \quad (4.3)$$

$$iW_x + \omega[S, W] = 0, \quad (4.4)$$

where $\omega = \text{const}$ and

$$Z = \frac{1}{4}([S, S_y] + 2iuS). \quad (4.5)$$

Here w is another scalar potential function.

4.1 Lax representation

As an integrable equation, the ML-III equation admits a Lax representation. It is given by

$$\Phi_x = U\Phi, \quad (4.6)$$

$$\Phi_t = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\Phi_y + V\Phi, \quad (4.7)$$

where

$$U = -i\lambda S, \quad (4.8)$$

$$V = 4\epsilon_2\lambda^2 Z + \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W \quad (4.9)$$

with

$$V_1 = wS + i\epsilon_2(S_{xy} + [S_x, Z]), \quad (4.10)$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \quad (4.11)$$

4.2 Reductions

Let us now consider some reductions of the ML-III equation (4.1)-(4.4).

4.2.1 Case I: $\epsilon_2 = 0$.

In this case, the ML-III equation reduces to the following principal chiral equation

$$iS_t + \frac{1}{\omega}[S, W] = 0, \quad (4.12)$$

$$iW_x + \omega[S, W] = 0. \quad (4.13)$$

As well-known this equation is integrable by IST.

4.2.2 Case II: $W = 0$.

In this case we get the following integrable equation

$$iS_t + i\epsilon_2(S_{xy} + [S_x, Z])_x + (wS)_x = 0, \quad (4.14)$$

$$u_x - \frac{i}{4}tr(S \cdot [S_x, S_y]) = 0, \quad (4.15)$$

$$w_x - \frac{i}{4}\epsilon_2[tr(S_x^2)]_y = 0. \quad (4.16)$$

4.2.3 Case III: $y = x$.

This case corresponds to the M-LXIV equation which reads as (see for e.g. [25])

$$iS_t + i\epsilon_2 S_{xxx} + \frac{i}{4}\epsilon_2(tr(S_x^2)S)_x + \frac{1}{\omega}[S, W] = 0, \quad (4.17)$$

$$iW_x + \omega[S, W] = 0. \quad (4.18)$$

4.3 Equivalent counterpart of the ML-III equation

It is not difficult to verify that the counterpart of the ML-III equation has the form

$$iq_t + i\epsilon_2 q_{xxy} - vq + (wq)_x - 2ip = 0, \quad (4.19)$$

$$ir_t + i\epsilon_2 r_{xxy} + vr + (wr)_x - 2ik = 0, \quad (4.20)$$

$$v_x - 2i\epsilon_2(r_{xy}q - rq_{xy}) = 0, \quad (4.21)$$

$$w_x - 2i\epsilon_2(rq)_y = 0, \quad (4.22)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (4.23)$$

$$k_x + 2i\omega k - 2\eta r = 0, \quad (4.24)$$

$$\eta_x + rp + kq = 0. \quad (4.25)$$

This equation can be considered as the (2+1)-dimensional complex modified Korteweg-de Vries-Maxwell-Bloch equation (cmKdVMBE) as it is one of (2+1)-dimensional generalizations of the (1+1)-dimensional cmKdVMBE (see e.g. [36]-[37]). Of course equation (4.19)-(4.25) is also integrable by IST. The corresponding Lax representation reads as

$$\Psi_x = A\Psi, \quad (4.26)$$

$$\Psi_t = 4\epsilon_2\lambda^2\Psi_y + B\Psi, \quad (4.27)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (4.28)$$

$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1}. \quad (4.29)$$

Here

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_{0y}, \quad (4.30)$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (4.31)$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & -\epsilon_2 q_{xy} + i\omega q \\ \epsilon_2 r_{xy} - i\omega r & 0 \end{pmatrix}, \quad (4.32)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \quad (4.33)$$

Now we assume that $r = \delta q^*$, $k = \delta p^*$. Then the system (4.19)-(4.25) takes the form

$$iq_t + i\epsilon_2 q_{xxy} - vq + (wq)_x - 2ip = 0, \quad (4.34)$$

$$v_x - 2i\epsilon_2\delta(q_{xy}^*q - q^*q_{xy}) = 0, \quad (4.35)$$

$$w_x - 2i\epsilon_2\delta(|q|^2)_y = 0, \quad (4.36)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (4.37)$$

$$\eta_x + \delta(q^*p + p^*q) = 0. \quad (4.38)$$

It is the (2+1)-dimensional cmKdVMBE. This equation admits the following integrable reduction, if $\epsilon_2 - 1 = p = \eta = 0$:

$$iq_t + iq_{xxy} - vq + (wq)_x = 0, \quad (4.39)$$

$$v_x - 2i\delta(q_{xy}^*q - q^*q_{xy}) = 0, \quad (4.40)$$

$$w_x - 2i\delta(|q|^2)_y = 0. \quad (4.41)$$

It is the usual (2+1)-dimensional cmKdV equation.

In (1+1) dimensions, that is if $y = x$, the cmKdVMBE (4.34)-(4.38) reduces to the (1+1)-dimensional cmKdVHMBE which has the form (see e.g. [36]-[37])

$$q_t + \epsilon_2(q_{xxx} + 6\delta|q|^2q_x) - 2p = 0, \quad (4.42)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (4.43)$$

$$\eta_x + \delta(q^*p + p^*q) = 0. \quad (4.44)$$

Its Lax representation reads as

$$\Psi_x = A\Psi, \quad (4.45)$$

$$\Psi_t = (4\epsilon_2\lambda^2 A + B)\Psi, \quad (4.46)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (4.47)$$

$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1}. \quad (4.48)$$

Here

$$B_1 = 2i\epsilon_2\delta|q|^2\sigma_3 + 2i\epsilon_2\sigma_3A_{0y}, \quad (4.49)$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (4.50)$$

$$B_0 = \epsilon_2\delta(q_x^*q - q^*q_x)\sigma_3 + B_{01}, \quad (4.51)$$

$$B_{01} = \begin{pmatrix} 0 & -\epsilon_2q_{xx} - 2\epsilon_2\delta|q|^2q \\ \epsilon_2r_{xx} + 2\epsilon_2\delta|q|^2r & 0 \end{pmatrix}, \quad (4.52)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \quad (4.53)$$

Note that the (1+1)-dimensional cmKdVMBE (4.42)-(4.44) admits the following integrable reductions.

i) The (1+1)-dimensional complex mKdV equation as $\epsilon_2 - 1 = p = \eta = 0$:

$$q_t + q_{xxx} + 6\delta|q|^2q_x = 0. \quad (4.54)$$

ii) The following (1+1)-dimensional equation when $\epsilon_2 = 0$:

$$q_t - 2p = 0, \quad (4.55)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (4.56)$$

$$\eta_x + \delta(q^*p + p^*q) = 0. \quad (4.57)$$

or

$$\frac{1}{2}q_{xt} - i\omega q_t - 2\eta q = 0, \quad (4.58)$$

$$2\eta_x + \delta(|q|^2)_t = 0. \quad (4.59)$$

iii) The following (1+1)-dimensional equation for $\delta = 0$:

$$q_t + \epsilon_2q_{xxx} - 2p = 0, \quad (4.60)$$

$$p_x - 2i\omega p - 2\eta_0q = 0, \quad (4.61)$$

where $\eta_0 = \text{const.}$ Again we note that all these reductions are integrable by IST. The corresponding Lax representations can be obtained from the Lax representation (4.45)-(4.46) as the corresponding reductions.

5 The Myrzakulov-Lakshmanan-IV equation

Our third new integrable spin system is the Myrzakulov-Lakshmanan-IV (ML-IV) equation, which has the form

$$iS_t + 2\epsilon_1Z_x + i\epsilon_2(S_{xy} + [S_x, Z])_x + (wS)_x + \frac{1}{\omega}[S, W] = 0, \quad (5.1)$$

$$u_x - \frac{i}{4}\text{tr}(S \cdot [S_x, S_y]) = 0, \quad (5.2)$$

$$w_x - \frac{i}{4}\epsilon_2[\text{tr}(S_x^2)]_y = 0, \quad (5.3)$$

$$iW_x + \omega[S, W] = 0, \quad (5.4)$$

where

$$Z = \frac{1}{4}([S, S_y] + 2iuS). \quad (5.5)$$

This equation is also integrable by the IST. Below we present some salient features on the ML-IV equation.

5.1 Lax representation

First let us we present the corresponding Lax representation of the ML-IV equation (5.1)-(5.4). It has the form

$$\Phi_x = U\Phi, \quad (5.6)$$

$$\Phi_t = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\Phi_y + V\Phi \quad (5.7)$$

with

$$U = -i\lambda S, \quad (5.8)$$

$$V = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)Z + \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W, \quad (5.9)$$

where

$$V_1 = wS + i\epsilon_2(S_{xy} + [S_x, Z]), \quad (5.10)$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}, \quad (5.11)$$

so that compatibility condition $\Phi_{xt} = \Phi_{tx}$ gives the LM-IV equation (5.1)-(5.4).

5.2 Reductions

Now we present some reductions of the LM-IV equation (5.1)-(5.4).

5.2.1 Case I: $\epsilon_1 = \epsilon_2 = 0$.

Let us we put $\epsilon_1 = \epsilon_2 = 0$. Then the LM-IV equation reduces to the form

$$iS_t + \frac{1}{\omega}[S, W] = 0, \quad (5.12)$$

$$iW_x + \omega[S, W] = 0. \quad (5.13)$$

It is nothing but the principal chiral equation noted previously, which is integrable.

5.2.2 Case II: $\epsilon_1 \neq 0, \epsilon_2 = 0$.

Next we consider the case $\epsilon_1 \neq 0, \epsilon_2 = 0$. Then we get

$$iS_t + 2\epsilon_1([S, S_y] + 2iuS)_x + \frac{1}{\omega}[S, W] = 0, \quad (5.14)$$

$$u_x - \frac{i}{4}tr(S \cdot [S_x, S_y]) = 0, \quad (5.15)$$

$$iW_x + \omega[S, W] = 0. \quad (5.16)$$

It is the ML-II equation (3.1)-(3.3).

5.2.3 Case III: $\epsilon_1 = 0, \epsilon_2 \neq 0$.

Our next example is the case $\epsilon_1 = 0, \epsilon_2 \neq 0$. In this case, the ML-IV equation takes the form

$$iS_t + i\epsilon_2(S_{xy} + [S_x, Z])_x + (wS)_x + \frac{1}{\omega}[S, W] = 0, \quad (5.17)$$

$$u_x - \frac{i}{4}tr(S \cdot [S_x, S_y]) = 0, \quad (5.18)$$

$$w_x - \frac{i}{4}\epsilon_2[tr(S_x^2)]_y = 0, \quad (5.19)$$

$$iW_x + \omega[S, W] = 0. \quad (5.20)$$

It is nothing but the ML-III equation (4.1)-(4.4).

5.2.4 Case IV: $W = 0$.

Now we put $W = 0$. Then we have

$$iS_t + 2\epsilon_1 Z_x + i\epsilon_2(S_{xy} + [S_x, Z])_x + (wS)_x = 0, \quad (5.21)$$

$$u_x - \frac{i}{4} \text{tr}(S \cdot [S_x, S_y]) = 0, \quad (5.22)$$

$$w_x - \frac{i}{4} \epsilon_2 [\text{tr}(S_x^2)]_y = 0. \quad (5.23)$$

5.2.5 Case V: $y = x$.

The last example is the case $y = x$, that is the (1+1)-dimensional case. This case corresponds to the equation

$$iS_t + \frac{1}{2} \epsilon_1 [S, S_{xx}] + i\epsilon_2(S_{xx} + 6\text{tr}(S_x^2)S)_x + \frac{1}{\omega} [S, W] = 0, \quad (5.24)$$

$$iW_x + \omega [S, W] = 0. \quad (5.25)$$

It is the M-XCIV equation (see e.g. [25]).

5.3 Equivalent counterpart of the ML-IV equation

The gauge equivalent counterpart of the ML-IV equation (5.1)-(5.4) has the form

$$iq_t + \epsilon_1 q_{xy} + i\epsilon_2 q_{xxy} - vq + (wq)_x - 2ip = 0, \quad (5.26)$$

$$ir_t - \epsilon_1 r_{xy} + i\epsilon_2 r_{xxy} + vr + (wr)_x - 2ik = 0, \quad (5.27)$$

$$v_x + 2\epsilon_1 (rq)_y - 2i\epsilon_2 (r_{xy}q - rq_{xy}) = 0, \quad (5.28)$$

$$w_x - 2i\epsilon_2 (rq)_y = 0, \quad (5.29)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.30)$$

$$k_x + 2i\omega k - 2\eta r = 0, \quad (5.31)$$

$$\eta_x + rp + kq = 0. \quad (5.32)$$

We designate this set of equations as the (2+1)-dimensional Hirota-Maxwell-Bloch equation (HMBE) for the reason that when $y = x$ it gives the (1+1)-dimensional Hirota-Maxwell-Bloch equation (HMBE) (see e.g. [36]-[37]). The set of equations (5.26)-(5.32) is also integrable by IST. The corresponding Lax representation reads as

$$\Psi_x = A\Psi, \quad (5.33)$$

$$\Psi_t = (2\epsilon_1 \lambda + 4\epsilon_2 \lambda^2) \Psi_y + B\Psi, \quad (5.34)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (5.35)$$

$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1}. \quad (5.36)$$

Here

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_{0y}, \quad (5.37)$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (5.38)$$

$$B_0 = -\frac{i}{2} v\sigma_3 + \begin{pmatrix} 0 & i\epsilon_1 q_y - \epsilon_2 q_{xy} + i\omega q \\ i\epsilon_1 r_y + \epsilon_2 r_{xy} - i\omega r & 0 \end{pmatrix}, \quad (5.39)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \quad (5.40)$$

Now we assume that $r = \delta q^*$, $k = \delta p^*$. Then the system (5.26)-(5.32) takes the form

$$iq_t + \epsilon_1 q_{xy} + i\epsilon_2 q_{xxy} - vq + (wq)_x - 2ip = 0, \quad (5.41)$$

$$v_x + 2\epsilon_1 \delta(|q|^2)_y - 2i\epsilon_2 \delta(q_{xy}^* q - q^* q_{xy}) = 0, \quad (5.42)$$

$$w_x - 2i\epsilon_2 \delta(|q|^2)_y = 0, \quad (5.43)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.44)$$

$$\eta_x + \delta(q^* p + p^* q) = 0. \quad (5.45)$$

It is the (2+1)-dimensional HMBE (compare with the HMBE from [36]-[37]). This equation admits the following integrable reductions.

i) For the case $\epsilon_1 - 1 = \epsilon_2 = p = \eta = 0$, we get

$$iq_t + q_{xy} + -vq = 0, \quad (5.46)$$

$$v_x + 2\delta(|q|^2)_y = 0 \quad (5.47)$$

which is the well-known (2+1)-dimensional nonlinear Schrödinger equation.

ii) The (2+1)-dimensional complex mKdV equation is obtained for the choice $\epsilon_1 = \epsilon_2 - 1 = p = \eta = 0$:

$$iq_t + i\epsilon_2 q_{xxy} - vq + (wq)_x = 0, \quad (5.48)$$

$$v_x - 2i\epsilon_2 \delta(q_{xy}^* q - q^* q_{xy}) = 0, \quad (5.49)$$

$$w_x - 2i\epsilon_2 \delta(|q|^2)_y = 0. \quad (5.50)$$

iii) The (2+1)-dimensional Schrodinger-Maxwell-Bloch equation results when $\epsilon_1 - 1 = \epsilon_2 = 0$:

$$iq_t + q_{xy} - vq + (wq)_x - 2ip = 0, \quad (5.51)$$

$$v_x + 2\delta(|q|^2)_y = 0, \quad (5.52)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.53)$$

$$\eta_x + \delta(q^* p + p^* q) = 0. \quad (5.54)$$

iv) The (2+1)-dimensional complex mKdV-Maxwell-Bloch equation is obtained for $\epsilon_1 = \epsilon_2 - 1 = 0$:

$$iq_t + iq_{xxy} - vq + (wq)_x - 2ip = 0, \quad (5.55)$$

$$v_x - 2i\delta(q_{xy}^* q - q^* q_{xy}) = 0, \quad (5.56)$$

$$w_x - 2i\delta(|q|^2)_y = 0, \quad (5.57)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.58)$$

$$\eta_x + \delta(q^* p + p^* q) = 0. \quad (5.59)$$

v) The following (1+1)-dimensional equation is obtained for $\epsilon_1 = \epsilon_2 = 0$:

$$q_t - 2p = 0, \quad (5.60)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.61)$$

$$\eta_x + \delta(q^* p + p^* q) = 0. \quad (5.62)$$

or

$$\frac{1}{2}q_{xt} - i\omega q_t - 2\eta q = 0, \quad (5.63)$$

$$2\eta_x + \delta(|q|^2)_t = 0. \quad (5.64)$$

vi) The following (1+1)-dimensional equation is obtained for $\delta = 0$:

$$iq_t + \epsilon_1 q_{xy} + i\epsilon_2 q_{xxy} - 2ip = 0, \quad (5.65)$$

$$p_x - 2i\omega p - 2\eta_0 q = 0, \quad (5.66)$$

where $\eta_0 = 0$. Again we note that all these reductions are integrable by IST. The corresponding Lax representations are obtained from the Lax representation (5.33)-(5.34) as the corresponding reductions. In (1+1) dimensions that is if $y = x$, this system reduces to the (1+1)-dimensional HMBE which has the form (see e.g. [36]-[37])

$$iq_t + \epsilon_1(q_{xx} + 2\delta|q|^2q) + i\epsilon_2(q_{xxx} + 6\delta|q|^2q_x) - 2ip = 0, \quad (5.67)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.68)$$

$$\eta_x + \delta(q^*p + p^*q) = 0. \quad (5.69)$$

Its Lax representation reads as

$$\Psi_x = A\Psi, \quad (5.70)$$

$$\Psi_t = ((2\epsilon_1\lambda + 4\epsilon_2\lambda^2)A + B)\Psi, \quad (5.71)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (5.72)$$

$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1}. \quad (5.73)$$

Here

$$B_1 = 2i\epsilon_2\delta|q|^2\sigma_3 + 2i\epsilon_2\sigma_3A_{0y}, \quad (5.74)$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (5.75)$$

$$B_0 = (i\epsilon_1\delta|q|^2 + \epsilon_2\delta(q_x^*q - q^*q_x))\sigma_3 + B_{01}, \quad (5.76)$$

$$B_{01} = \begin{pmatrix} 0 & i\epsilon_1q_x - \epsilon_2q_{xx} - 2\epsilon_2\delta|q|^2q \\ i\epsilon_1r_x + \epsilon_2r_{xx} + 2\epsilon_2\delta|q|^2r & 0 \end{pmatrix}, \quad (5.77)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \quad (5.78)$$

Note that the (1+1)-dimensional HMBE (5.67)-(5.69) admits the following integrable reductions.

i) The NSLE for $\epsilon_1 - 1 = \epsilon_2 = p = \eta = 0$:

$$iq_t + q_{xx} + 2\delta|q|^2q = 0. \quad (5.79)$$

ii) The (1+1)-dimensional complex mKdV equation for $\epsilon_1 = \epsilon_2 - 1 = p = \eta = 0$:

$$q_t + q_{xxx} + 6\delta|q|^2q_x = 0. \quad (5.80)$$

iii) The (1+1)-dimensional Schrodinger-Maxwell-Bloch equation for $\epsilon_1 - 1 = \epsilon_2 = 0$:

$$iq_t + q_{xx} + 2\delta|q|^2q - 2ip = 0, \quad (5.81)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.82)$$

$$\eta_x + \delta(q^*p + p^*q) = 0. \quad (5.83)$$

iv) The (1+1)-dimensional complex mKdV-Maxwell-Bloch equation for $\epsilon_1 = \epsilon_2 - 1 = 0$:

$$q_t + q_{xxx} + 6\delta|q|^2q_x - 2p = 0, \quad (5.84)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.85)$$

$$\eta_x + \delta(q^*p + p^*q) = 0. \quad (5.86)$$

v) The following (1+1)-dimensional equation is obtained for $\epsilon_1 = \epsilon_2 = 0$:

$$q_t - 2p = 0, \quad (5.87)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.88)$$

$$\eta_x + \delta(q^*p + p^*q) = 0. \quad (5.89)$$

or

$$\frac{1}{2}q_{xt} - i\omega q_t - 2\eta q = 0, \quad (5.90)$$

$$2\eta_x + \delta(|q|^2)_t = 0. \quad (5.91)$$

vi) The following (1+1)-dimensional equation is obtained for $\delta = 0$:

$$iq_t + \epsilon_1 q_{xx} + i\epsilon_2 q_{xxx} - 2ip = 0, \quad (5.92)$$

$$p_x - 2i\omega p - 2\eta_0 q = 0, \quad (5.93)$$

where $\eta_0 = 0$. Again we note that all these reductions are integrable by IST. The corresponding Lax representations can be got from the Lax representation (5.70)-(5.71) as appropriate reductions.

6 Conclusion

Spin systems are fascinating nonlinear dynamical systems. In particular integrable spin systems have much relevance in applied ferromagnetism and nanomagnetism. More interestingly, integrable spin systems have close connection with nonlinear Schrödinger family of equations. In this paper, we have introduced three specific cases of (2+1) dimensional integrable spin systems, which we designated as Myrzakulov-Lakshmanan II, III and IV equations, where additional scalar potentials or vector potentials interact in specific ways with the spin fields. Through appropriate gauge or geometric equivalence we have identified the three equivalent (2+1) dimensional nonlinear Schrödinger family of equations along with their Lax pairs. Regarding these equations, an extremely interesting question is to investigate what are the physical applications these new equations, for example in nonlinear optics (for their (1+1)-dimensional analogues, see for e.g. Refs. [34]-[37]). It will be interesting to investigate the associated other integrability properties like infinite number of conservation laws, involutive integrals of motion, soliton solutions, exponentially localized dromion solutions, etc. In particular, it is interesting to investigate the relation between the above presented integrable spin systems with self-consistent potentials and geometry of curves and surfaces (see, e.g. Refs. [14]-[33]). Work is in progress along these lines and the results will be reported separately.

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